2023.08.31　第19回 HPC-Phys 勉強会

ラグランジュ的定式化を用いた 2次元の星の平衡形状の新たな計算法

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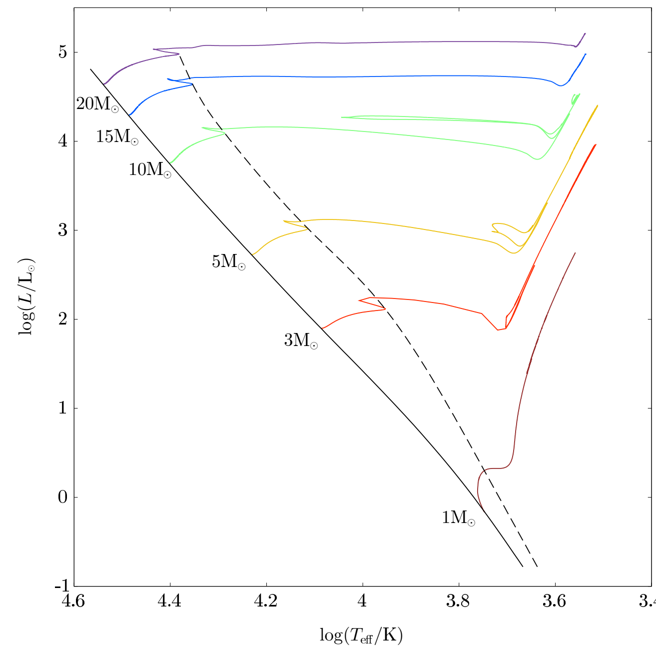
Contents

- Introduction
- Numerical Methods
  - Configuration (Lagrangian formulation)
  - Self-gravity (Spectral method)
- Results
- Summary

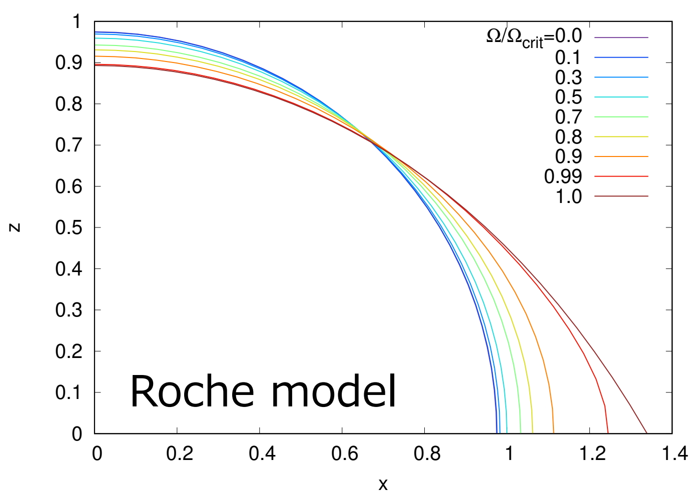
Introduction

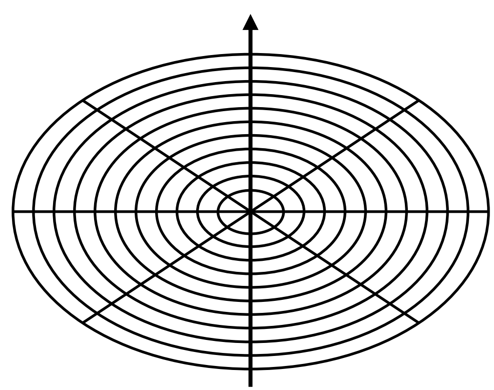
■ 1D stellar evolution calculation

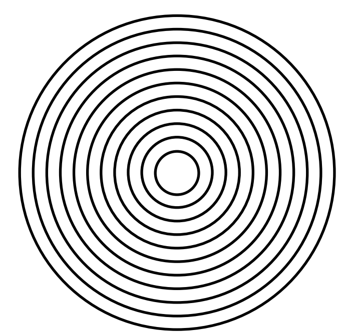
Basic calculation
to understand stellar evolution
(Henyey et al. 1959)



✓ Angle dependence averaged



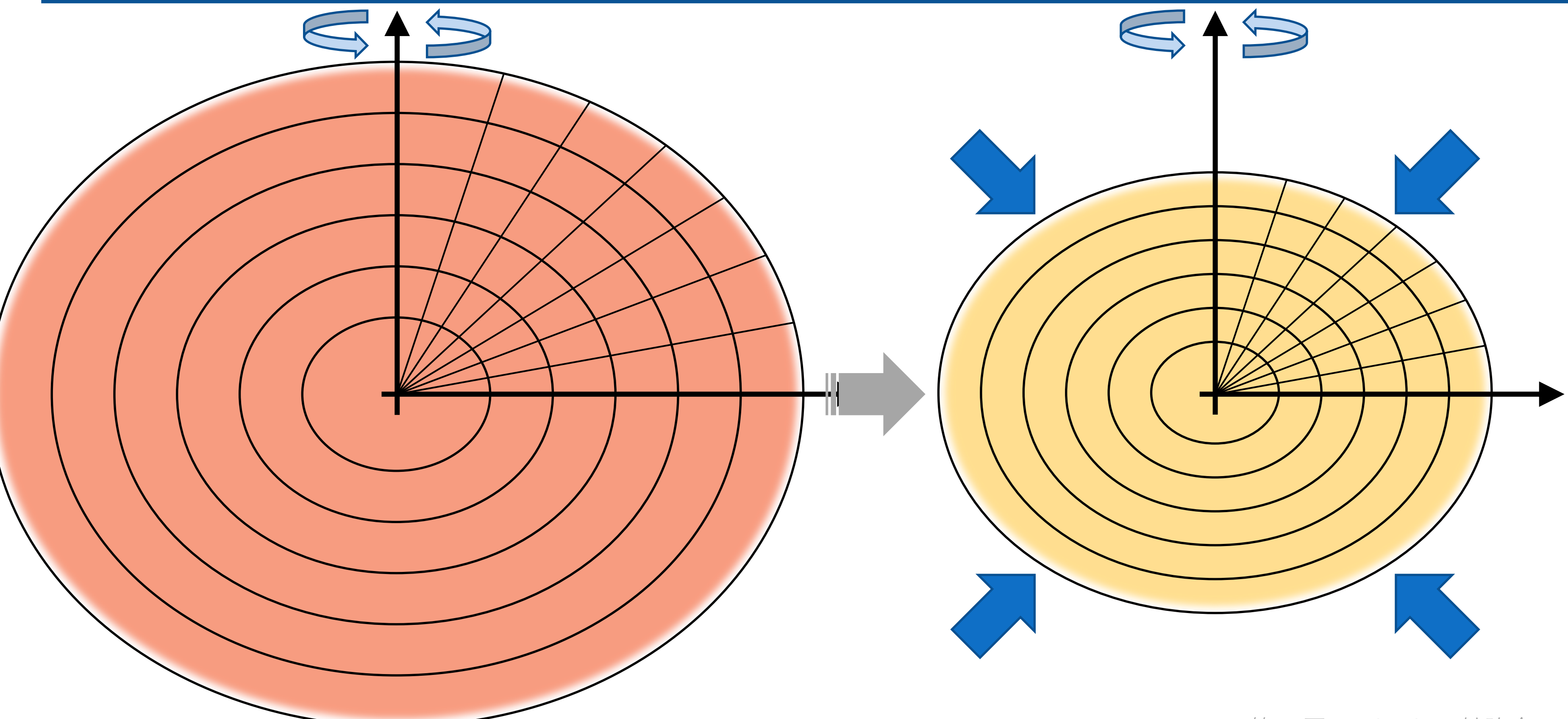




Introduction

Calculations correctly evaluate the effect of rotation

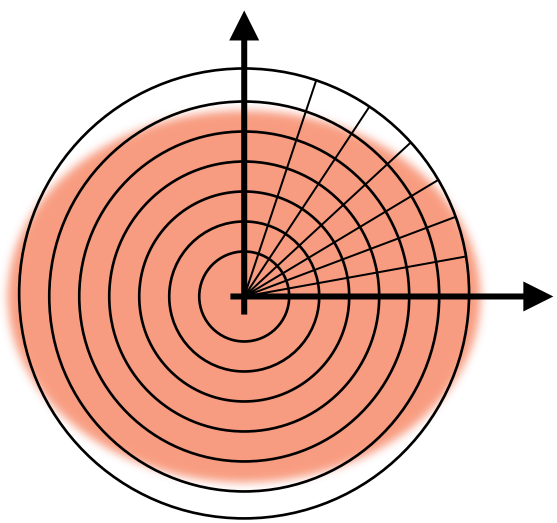
- All previous stellar evolution calculations are **1D.**
- No 2D structure calculations using **Lagrangian coordinate.**



Introduction

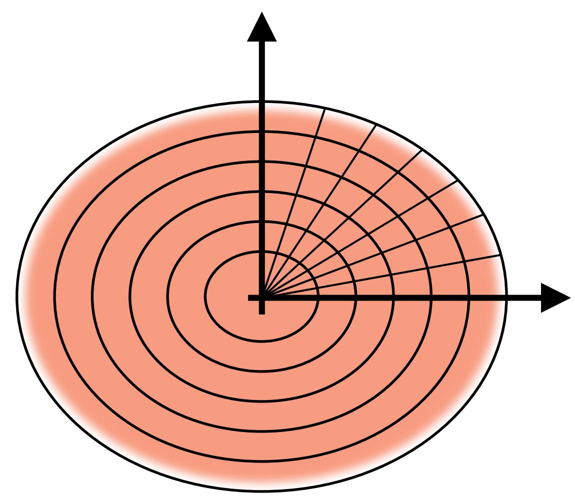
■ Eulerian/Lagrangian coordinates

- **Eulerian :** Coordinates fixed in space
  Solve for the value of a physical quantity on coordinate
- **Lagrangian :** Coordinates fixed to a fluid element
  Solve for where the physical quantity moves to



2D Eulerian coordinate

Solve for the value of a physical quantity



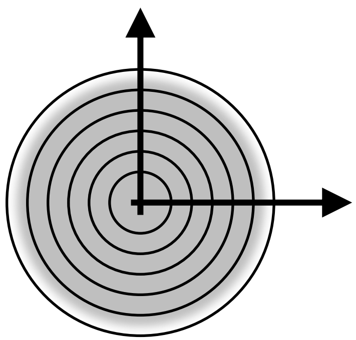
2D Lagrangian coordinate

Solve for the position

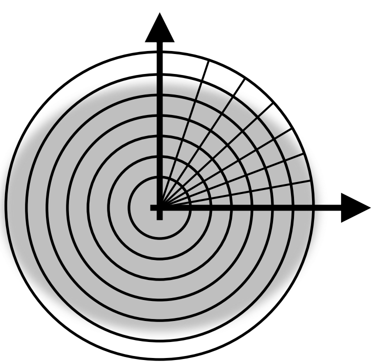
Introduction

- ✓ Suitable coordinates for the expansion/contraction of stars
- ✓ Appropriate introduction of rotation effects

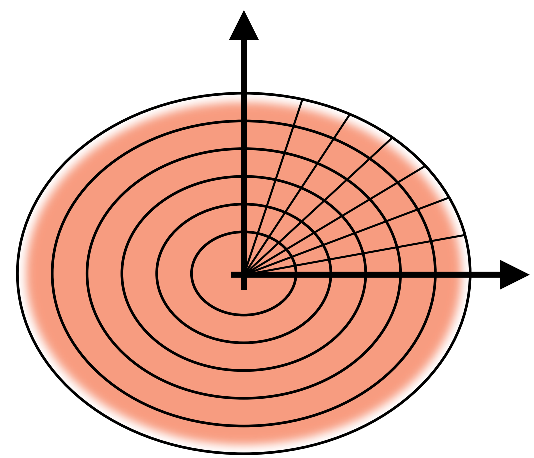
➡ **2D Lagrangian coordinate**



1D Lagrangian coordinate



2D Eulerian coordinate



2D Lagrangian coordinate

Introduction

■ Flow of evolution calculation

Dynamical timescale $\ll$ Evolution timescale

Ⅰ. Hydrostatic equilibrium calculation

Ⅱ. Time evolution calculation

Nuclear network

Energy transportation

Repeat ➡ Evolution Calculation

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  - ✓ Configuration (Lagrangian formulation)
  - ✓ Self-gravity (Spectral method)
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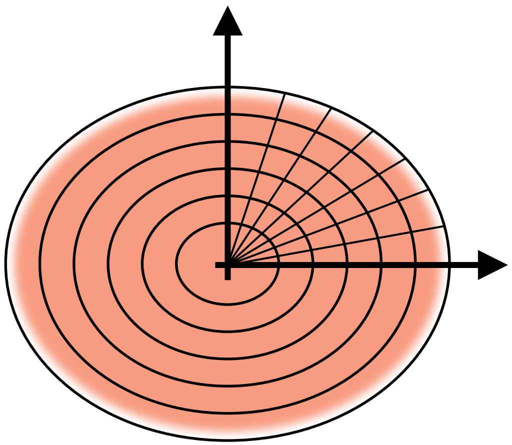
Flow of calculation



Configuration

- ✓Lagrangian formulation
- ✓W4 method

Self gravity

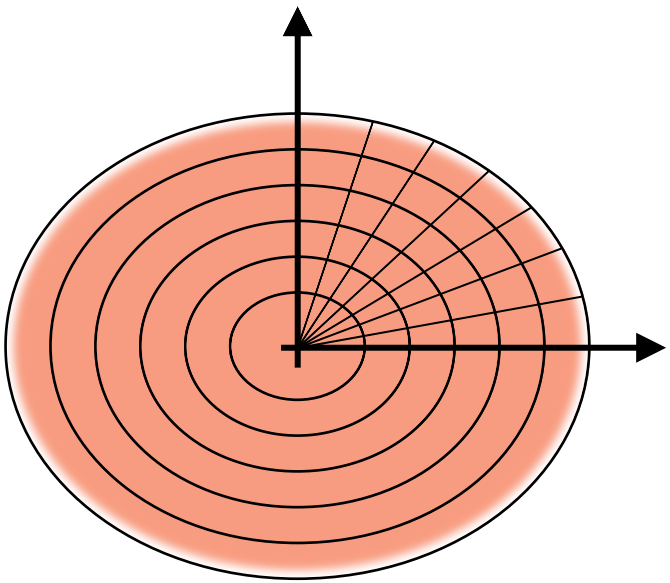
- ✓ Spectral method

Remeshing

- ✓ Interpolation

Method: Configuration

■Lagrangian formulation



Force-balance eq. :

$$\frac{1}{\rho}\nabla \mathrm{P} = -\nabla\phi + \frac{1}{2}\Omega^2\nabla(R\sin\Theta)^2$$

Solve for the position
(Not for values of physical quantities)
↓
If not set up appropriately,
no solution exists.

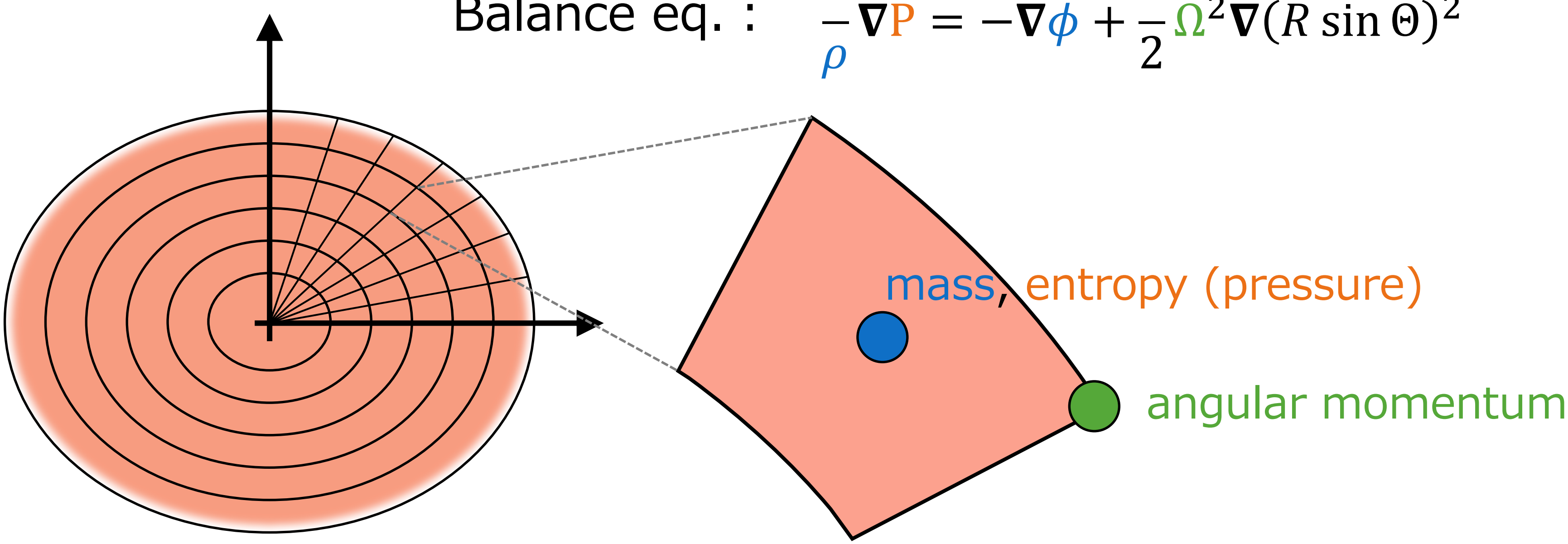
Assumption: axisymmetry, equatorial symmetry

Method: Configuration

■ Flow of configuration calculation

1. Setting distributions of mass, entropy and angular momentum
2. Finding the hydrostatic equilibrium configuration without changing the given physical quantities

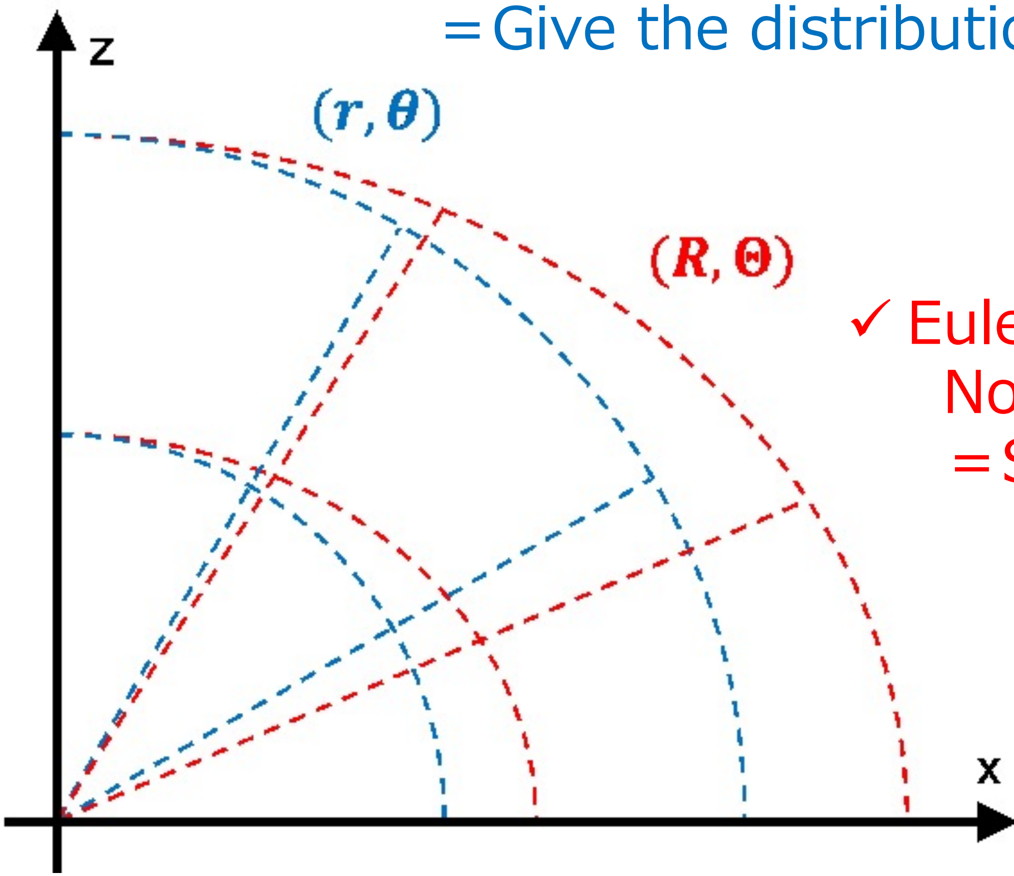
Balance eq. : $$\frac{1}{\rho}\nabla P = -\nabla\phi + \frac{1}{2}\Omega^2\nabla(R\sin\Theta)^2$$



Method: Configuration

- Two types of coordinates

✓ Lagrangian coordinate
Spherical reference configuration
＝Give the distribution of physical quantities

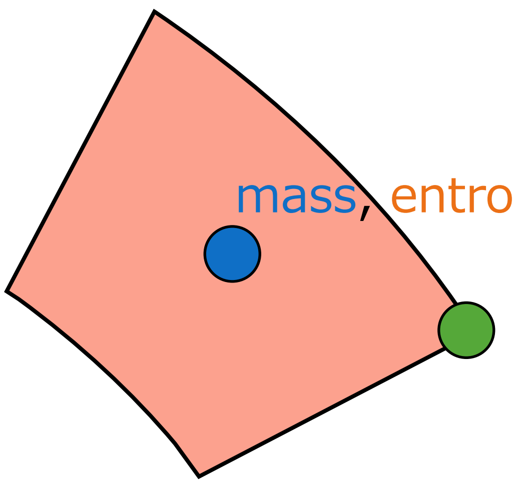


✓ Eulerian coordinate
Non-spherical configuration
＝Spatial position

Method: Configuration

■ Flow of configuration calculation

1. Setting distributions of mass, entropy and angular momentum
2. Finding the hydrostatic equilibrium configuration without changing the given physical quantities



Balance eq. : $$\frac{1}{\rho}\nabla \mathrm{P} = -\nabla\phi + \frac{1}{2}\Omega^2\nabla(R\sin\Theta)^2$$

Density: $$\rho = \frac{\Delta m}{\Delta V}$$

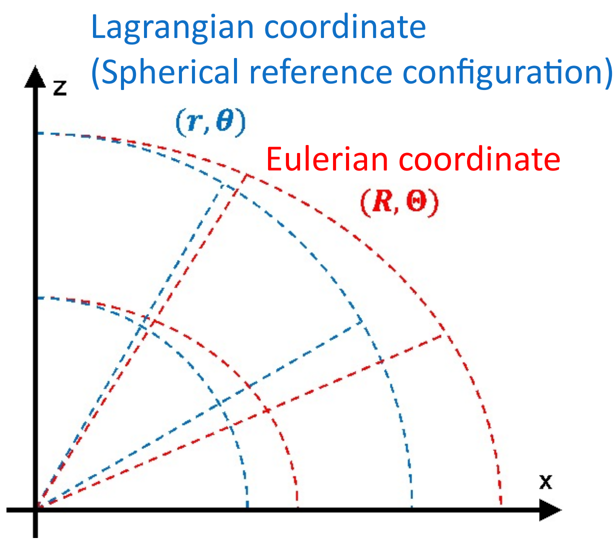
$$\Delta V = |\det J|\Delta v$$

$$J = \begin{bmatrix} \left.\frac{\partial R^3/3}{\partial r^3/3}\right|_{\cos\theta} & \left.\frac{\partial \cos\Theta}{\partial r^3/3}\right|_{\cos\theta} \\ \left.\frac{\partial R^3/3}{\partial \cos\theta}\right|_{r^3/3} & \left.\frac{\partial \cos\Theta}{\partial \cos\theta}\right|_{r^3/3} \end{bmatrix}$$

EOS: $P = P(\rho, s)$

Rotation law: $\Omega = \Omega(R, \Theta)$

Method: Configuration



$$\frac{1}{\rho}\nabla P + \nabla\phi - \frac{1}{2}\Omega^2\nabla(R\sin\Theta)^2 = 0$$: Balance eq.

$$\begin{cases} F_R = \dfrac{\partial P}{\partial R} + \rho\dfrac{\partial\phi}{\partial R} - \rho(R\sin\Theta)\Omega^2\sin\Theta = 0 \\ F_\Theta = \dfrac{\partial P}{R\partial\Theta} + \rho\dfrac{\partial\phi}{R\partial\Theta} - \rho(R\sin\Theta)\Omega^2\cos\Theta = 0 \end{cases}$$

$$\begin{pmatrix} F_r \\ F_\theta \end{pmatrix} = \begin{pmatrix} \dfrac{\partial R}{\partial r} & \dfrac{\mathrm{R}\partial\Theta}{\partial r} \\ \dfrac{\partial R}{r\partial\theta} & \dfrac{R\partial\Theta}{r\partial\theta} \end{pmatrix} \begin{pmatrix} F_R \\ F_\Theta \end{pmatrix}$$: coordinate $(R, \Theta) \to (r, \theta)$

$$F_r = \frac{dP}{dr} + \rho\left[\left(\frac{\partial R}{\partial r}\right)F_{\mathrm{grav}}^{(R)} + \left(\frac{R\partial\Theta}{\partial r}\right)F_{\mathrm{grav}}^{(\Theta)}\right] + \frac{\rho j^2}{(R\sin\Theta)^3}\left[\left(\frac{\partial R}{\partial r}\right)\sin\Theta + \left(\frac{R\partial\Theta}{\partial r}\right)\cos\Theta\right] = 0$$

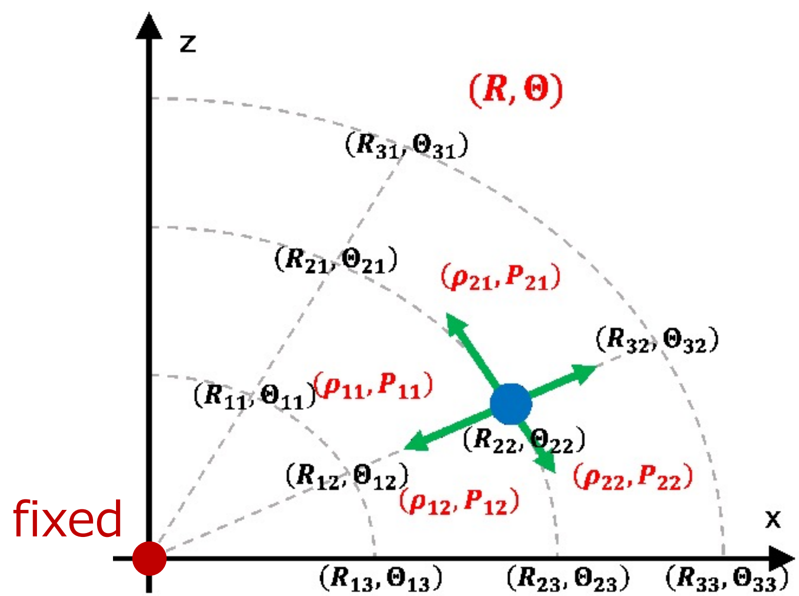
$$F_\theta = \frac{dP}{rd\theta} + \rho\left[\left(\frac{\partial R}{r\partial\theta}\right)F_{\mathrm{grav}}^{(R)} + \left(\frac{R\partial\Theta}{r\partial\theta}\right)F_{\mathrm{grav}}^{(\Theta)}\right] + \frac{\rho j^2}{(R\sin\Theta)^3}\left[\left(\frac{\partial R}{r\partial\theta}\right)\sin\Theta + \left(\frac{R\partial\Theta}{r\partial\theta}\right)\cos\Theta\right] = 0$$

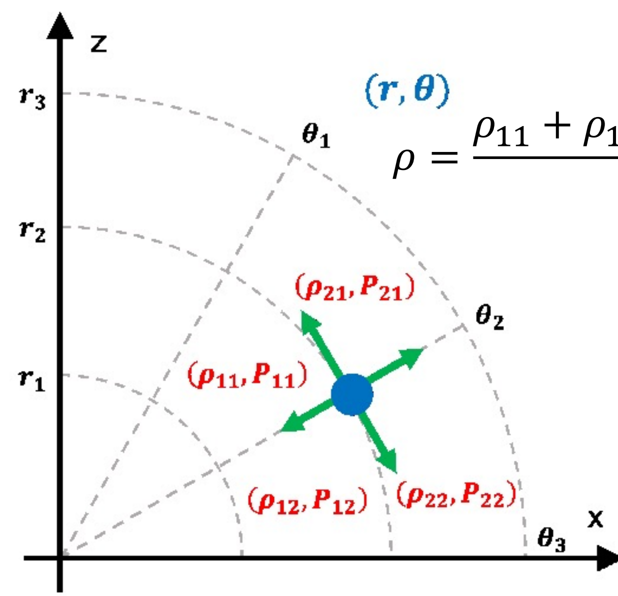
Method: Configuration

Differentiation :

$$F_r = \frac{P_{21} + P_{22} - P_{11} - P_{12}}{r_3 - r_1} + \rho \left[\left(\frac{R_{32} - R_{12}}{r_3 - r_1} \right) F_{\text{grav}}^{(R)} + \left(\frac{R_{22}(\Theta_{32} - \Theta_{12})}{r_3 - r_1} \right) F_{\text{grav}}^{(\Theta)} \right] + \frac{\rho j^2}{(R_{22} \sin \Theta_{22})^3} \left[\left(\frac{R_{23} - R_{21}}{r_3 - r_1} \right) \sin \Theta_{22} + \left(\frac{R_{22}(\Theta_{32} - \Theta_{12})}{r_3 - r_1} \right) \cos \Theta_{22} \right] = 0$$

$$F_\theta = \frac{P_{12} + P_{22} - P_{11} - P_{21}}{r_2(\theta_3 - \theta_1)} + \rho \left[\left(\frac{R_{23} - R_{21}}{r_2(\theta_3 - \theta_1)} \right) F_{\text{grav}}^{(R)} + \left(\frac{R_{22}(\Theta_{23} - \Theta_{21})}{r_2(\theta_3 - \theta_1)} \right) F_{\text{grav}}^{(\Theta)} \right] + \frac{\rho j^2}{(R_{22} \sin \Theta_{22})^3} \left[\left(\frac{R_{23} - R_{21}}{r_2(\theta_3 - \theta_1)} \right) \sin \Theta_{22} + \left(\frac{R_{22}(\Theta_{23} - \Theta_{21})}{r_2(\theta_3 - \theta_1)} \right) \cos \Theta_{22} \right] = 0$$





$$\rho = \frac{\rho_{11} + \rho_{12} + \rho_{21} + \rho_{22}}{4}$$

Method: Configuration

■ W4 method   Okawa et al. 2023

Root-finding scheme for nonlinear simultaneous equations using iterative methods

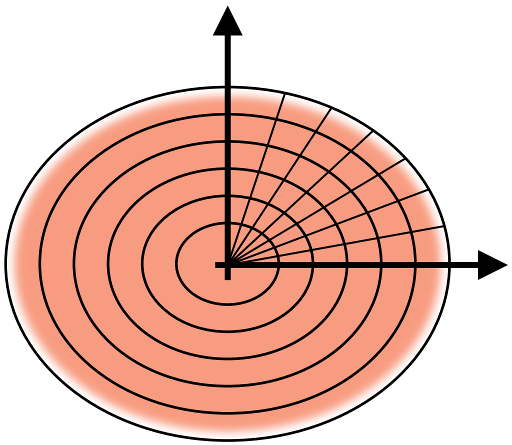
- Local convergence similar to the Newton method
- Excellent global convergence

→ Useful for stiff equations which cannot be solved by the Newton method

$$\frac{d^2\boldsymbol{x}}{d\tau^2} + M_1 \frac{d\boldsymbol{x}}{d\tau} + M_2 \boldsymbol{F}(\boldsymbol{x}) = 0 \quad \Longrightarrow \quad \begin{cases} \dfrac{d\boldsymbol{x}}{d\tau} = X\boldsymbol{p} \\ \dfrac{d\boldsymbol{p}}{d\tau} = -2\boldsymbol{p} - Y\boldsymbol{F} \end{cases}$$

$$\begin{cases} \boldsymbol{x}_{n+1} = \boldsymbol{x}_n + \Delta\tau X \boldsymbol{p}_n \\ \boldsymbol{p}_{n+1} = (1 - 2\Delta\tau)\boldsymbol{p}_n - \Delta\tau Y \boldsymbol{F}(\boldsymbol{x}_n) \end{cases}$$

Flow of calculation



Configuration

- ✓Lagrangian formulation
- ✓W4 method

Self gravity

- ✓ Spectral method

Remeshing

- ✓ Interpolation

Method: Self-gravity

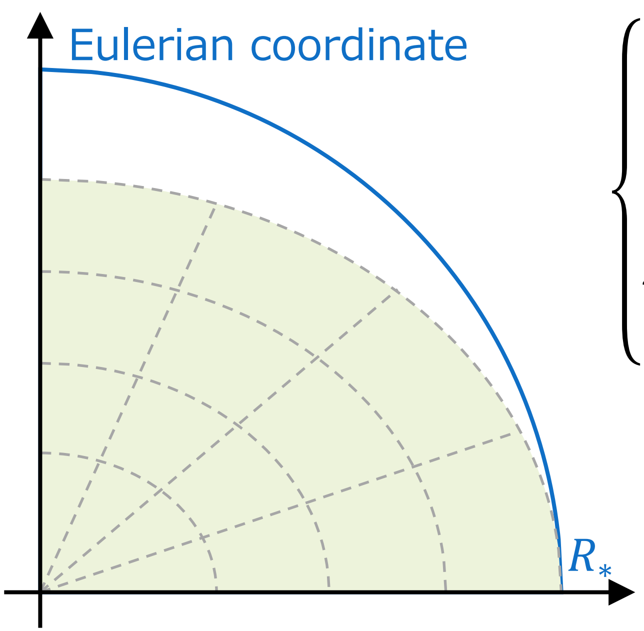
- Solve Poisson's equation by spectral method with Legendre polynomial $(\mu = \cos\Theta)$

Use the Eulerian coordinate

✓ Laplacian takes the simplest form

✓ Application of the spectral method becomes easiest

$$\nabla^2\phi(R,\mu) = 4\pi G\rho(R,\mu) \rightarrow \frac{\partial}{\partial R}\left(R^2\frac{\partial\phi}{\partial R}\right) + \frac{\partial}{\partial\mu}\left[(1-\mu^2)\frac{\partial\phi}{\partial\mu}\right] = 4\pi G\rho R^2$$



$$\begin{cases} \phi(R,\mu) = \sum_{l=0}^{l_{\max}}\sum_{m=0}^{m_{\max}} \boldsymbol{a_{lm}}\tilde{P}_l(R)P_m(\mu) \\ 4\pi G\rho(R,\mu)R^2 = \sum_{l=0}^{l_{\max}}\sum_{m=0}^{m_{\max}} \boldsymbol{b_{lm}}\tilde{P}_l(R)P_m(\mu) \end{cases}$$

P_m: Legendre polynomial

$\tilde{P}_l$: shifted Legendre polynomial

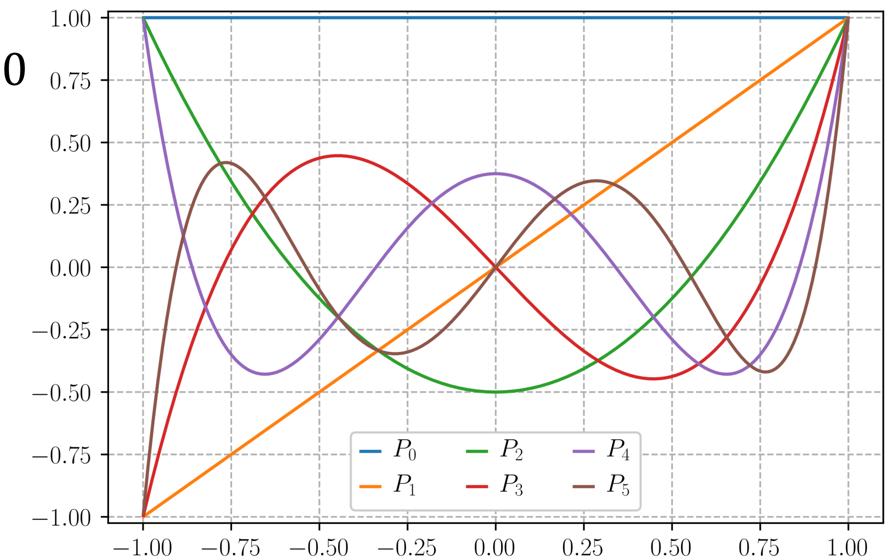
Method: Self-gravity

- Solve Poisson's equation by spectral method with Legendre polynomial

$$\nabla^2\phi(R,\mu) = 4\pi G\rho(R,\mu) \;\rightarrow\; \frac{\partial}{\partial R}\left(R^2\frac{\partial\phi}{\partial R}\right) + \frac{\partial}{\partial\mu}\left[(1-\mu^2)\frac{\partial\phi}{\partial\mu}\right] = 4\pi G\rho R^2$$

Legendre polynomial

$$\frac{d}{dx}\left[(1-x^2)\frac{d}{dx}f(x)\right] + \lambda(\lambda+1)f(x) = 0$$



Shifted Legendre polynomial

$$[-1,1] \rightarrow [0,R_*] \qquad R = R_*\frac{x+1}{2}$$

$$\int_0^{R_*}\tilde{P}_l(R)\tilde{P}_m(R)dR = \frac{R_*}{2l+1}\delta_{lm}$$

$$\frac{d}{dR}\tilde{P}_l(R) = \frac{2}{\Delta R}\frac{d}{dx}P_l(x)$$

Method: Self-gravity

$$\frac{\partial}{\partial R}\left(R^2\frac{\partial\phi}{\partial R}\right)+\frac{\partial}{\partial\mu}\left[(1-\mu^2)\frac{\partial\phi}{\partial\mu}\right]=4\pi G\rho(R,\mu)R^2$$

$$\begin{cases}\phi(R,\mu)=\sum_{l=0}^{l_{\max}}\sum_{m=0}^{m_{\max}}a_{lm}\widetilde{P}_l(R)P_m(\mu)\\ 4\pi G\rho(R,\mu)R^2=\sum_{l=0}^{l_{\max}}\sum_{m=0}^{m_{\max}}b_{lm}\widetilde{P}_l(R)P_m(\mu)\end{cases}$$

$$\sum\sum\left[\frac{d}{dR}\left(R^2\frac{d}{dR}\widetilde{P}_l(R)\right)a_{lm}P_m(\mu)-m(m+1)a_{lm}\,\widetilde{P}_l(R)P_m(\mu)\right]=\sum b_{lm}\widetilde{P}_l(R)P_m(\mu)$$

$$\sum\left[\frac{d}{dR}\left(R^2\frac{d}{dR}\widetilde{P}_l(R)\right)a_{lm}-m(m+1)a_{lm}\,\widetilde{P}_l(R)\right]=\sum b_{lm}\widetilde{P}_l(R)$$

Method: Self-gravity

- Differentiation of Legendre polynomial

$$\frac{d}{dR}\tilde{P}_l(R) = \frac{2}{\Delta R}\frac{d}{dx}P_l(x) = \sum C_{ll'}\tilde{P}_{l'}(R)$$

$$\frac{d}{dx}P_l(x) = (2l-1)P_{l-1}(x) + (2l-5)P_{l-3}(x) + \cdots$$

$$C_{ll'} = \frac{2}{\Delta R}\begin{pmatrix} 0 & 1 & 0 & 1 & 0 & 1 & \cdots & \\ & 0 & 3 & 0 & 3 & 0 & \cdots & \\ & & 0 & 5 & 0 & 5 & \cdots & \\ & & & & & & & \vdots \\ & & & & \ddots & & & 0 \\ & & & & & & & 2l-5 \\ & & & & & & & 0 \\ & 0 & & & & & & 2l-1 \\ & & & & & & & 0 \end{pmatrix}$$

Method: Self-gravity

- Multiplying Legendre polynomial by R^2

(Arfken 1985, p. 700)

$$R^2\widetilde{P_l}(R) = \sum D_{ln}\widetilde{P_n}(R)$$

$$D_{ln} = \frac{2n+1}{\Delta R}\left(\frac{\Delta R}{2}\right)^3\left[\frac{a^2}{2n+1}\int_{-1}^{1}P_l(x)P_n(x)dx + 2a\int_{-1}^{1}xP_l(x)P_n(x)dx + \int_{-1}^{1}x^2P_l(x)P_n(x)dx\right]$$

$$= \frac{2n+1}{\Delta R}\left(\frac{\Delta R}{2}\right)^3\left[\frac{a^2}{2n+1}\delta_{ln} + 2aD_{ln}^{(1)} + D_{ln}^{(2)}\right] \qquad a = \frac{R_{\rm in}+R_{\rm out}}{\Delta R}$$

$$D_{ln}^{(1)}: \qquad \int_{-1}^{1}xP_l(x)P_n(x)dx = \begin{cases} \dfrac{2(l+1)}{(2l+1)(2l+3)} & \text{for } n = l+1 \\ \dfrac{2l}{(2l-1)(2l+1)} & \text{for } n = l-1 \end{cases}$$

$$D_{ln}^{(2)}: \quad \int_{-1}^{1}x^2P_l(x)P_n(x)dx = \begin{cases} \dfrac{2(l+1)(l+2)}{(2l+1)(2l+3)(2l+5)} & \text{for } n = l+2 \\ \dfrac{2(2l^2+2l-1)}{(2l-1)(2l+1)(2l+3)} & \text{for } n = l \\ \dfrac{2l(l-1)}{(2l-3)(2l-1)(2l+1)} & \text{for } n = l-2 \end{cases}$$

Method: Self-gravity

$$\sum\left[\frac{\partial}{\partial R}\left(R^2\frac{\partial}{\partial R}\widetilde{P}_l(R)\right)a_{lm} - m(m+1)a_{lm}\,\widetilde{P}_l(R)\right] = \sum b_{lm}\widetilde{P}_l(R)$$

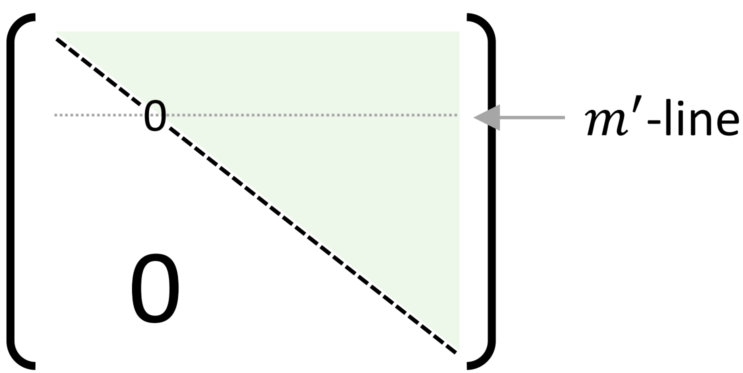
$$\sum\sum\left[\sum\sum C_{ll'''}D_{l'''l''}C_{l''l'} - m(m+1)\delta_{ll'}\right]a_{l'm}\widetilde{P}_l(R) = \sum b_{lm}\widetilde{P}_l(R)$$

$$= \boldsymbol{M'_{ll'}}$$

$$\sum\sum M'_{ll'}a_{l'm'}\widetilde{P}_l(R) = \sum b_{lm'}\widetilde{P}_l(R) \rightarrow \sum M'_{ll'}a_{l'm'} = b_{lm'}$$

$$a_{l'm'} = M^{-1}_{ll'}b_{lm'}$$

$$\boldsymbol{M'_{ll'}} =$$



Method: Self-gravity

- Boundary conditions

$$\phi_{m'} = \sum_l a_{lm'} \tilde{P}_l(R)$$

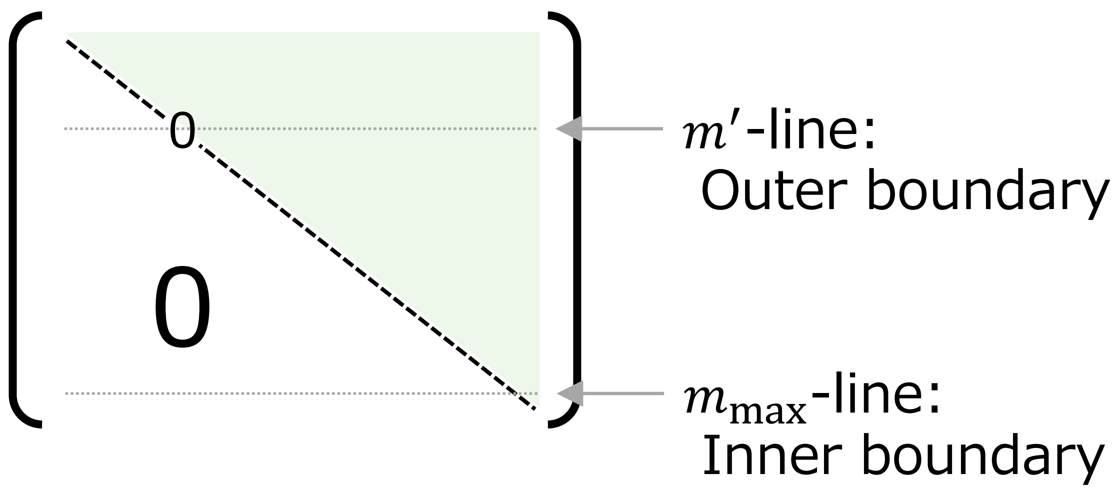
✓ Inner boundary

$$\left.\frac{\partial \phi_{m'}}{\partial R}\right|_{R=0} = 0 \qquad \Rightarrow \sum_l a_{lm'} \times (-1)^l l(l+1) = 0$$

✓ Outer boundary

$$\left.\frac{\partial \phi_{m'}}{\partial R}\right|_{R=R_*} = -\frac{m'+1}{R_*}\phi_{m'} \qquad \Rightarrow \sum_l a_{lm'} \times \{(m'+1)+(l-1)l\} = 0$$

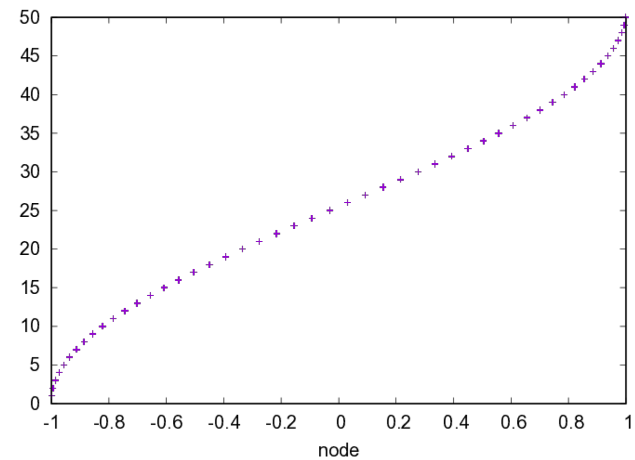
$\boldsymbol{M}'_{ll'} =$



Method: Self-gravity

- Coefficient of density

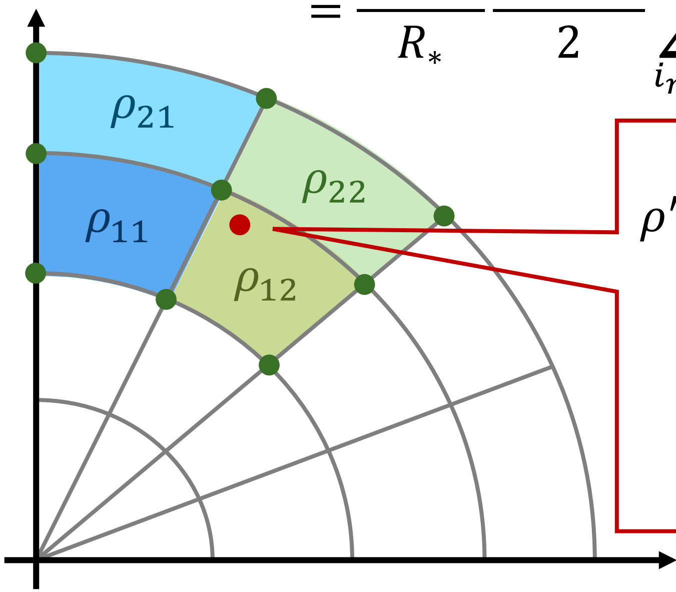
$$4\pi G\rho(R,\mu)R^2 = \sum_{l=0}^{l_{\max}}\sum_{m=0}^{m_{\max}} \boldsymbol{b_{lm}}\widetilde{P}_l(R)P_m(\mu)$$



✓ Gaussian quadrature method

$$b_{lm} = \frac{2l+1}{R_*}\frac{2m+1}{2}\int_0^{R_*}\int_{-1}^{1} 4\pi G\rho\,(R,\mu)R^2\widetilde{P}_l(R)P_m(\mu)dRd\mu$$

$$= \frac{2l+1}{R_*}\frac{2m+1}{2}\sum_{i_r=1}^{n_r}\sum_{i_\theta=1}^{n_\theta} 4\pi G\rho\left(x_{i_r},y_{i_\theta}\right)x_{i_r}^2\widetilde{P}_l\left(R\left(x_{i_r}\right)\right)P_m\left(y_{i_\theta}\right)w_{i_r}w_{i_\theta}$$



$$\rho'(r',\theta') = \sum_{j=1}^{3}\sum_{k=1}^{3}\widehat{M}_j(\alpha)\widehat{M}_k(\beta)\rho\left(r_{jk},\theta_{jk}\right) \quad (-1 \le \alpha,\beta \le 1)$$

$$\widehat{M}_1(\alpha) = -\frac{\alpha}{2}(1-\alpha)$$

$$\widehat{M}_2(\alpha) = (1+\alpha)(1-\alpha)$$

$$\widehat{M}_3(\alpha) = \frac{\alpha}{2}(1+\alpha)$$

Method: Self-gravity

- Expansion coefficients of gravitational force

$$a_{l'm'} = M_{ll'}^{-1} b_{lm'}$$

↓

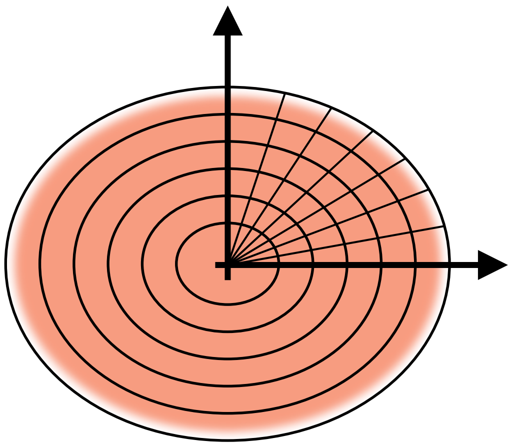
$$F_{\mathrm{grav}}^{(R)}(R,\mu) = \frac{\partial \phi}{\partial R} = \sum\sum\sum C_{ll'} a_{l'm} \tilde{P}_l(R) P_m(\mu)$$

$$= \sum_{l=0}^{l_{\max}-1} \sum_{m=0}^{m_{\max}} \boldsymbol{G_{lm}^{(R)}} \tilde{P}_l(R) P_m(\mu)$$

$$F_{\mathrm{grav}}^{(\Theta)}(R,\mu) = \frac{\partial \phi}{R \partial \Theta} = \frac{\sin\Theta}{R} \frac{\partial \phi}{\partial \cos\Theta} = \sum\sum\sum a_{lm'} C_{m'm} \tilde{P}_l(R) P_m(\mu)$$

$$= \sum_{l=0}^{l_{\max}} \sum_{m=0}^{m_{\max}-1} \boldsymbol{G_{lm}^{(\Theta)}} \tilde{P}_l(R) P_m(\mu)$$

Flow of calculation



Configuration

✓Lagrangian formulation

✓W4 method

↓

Self gravity

✓ Spectral method

↓

Remeshing

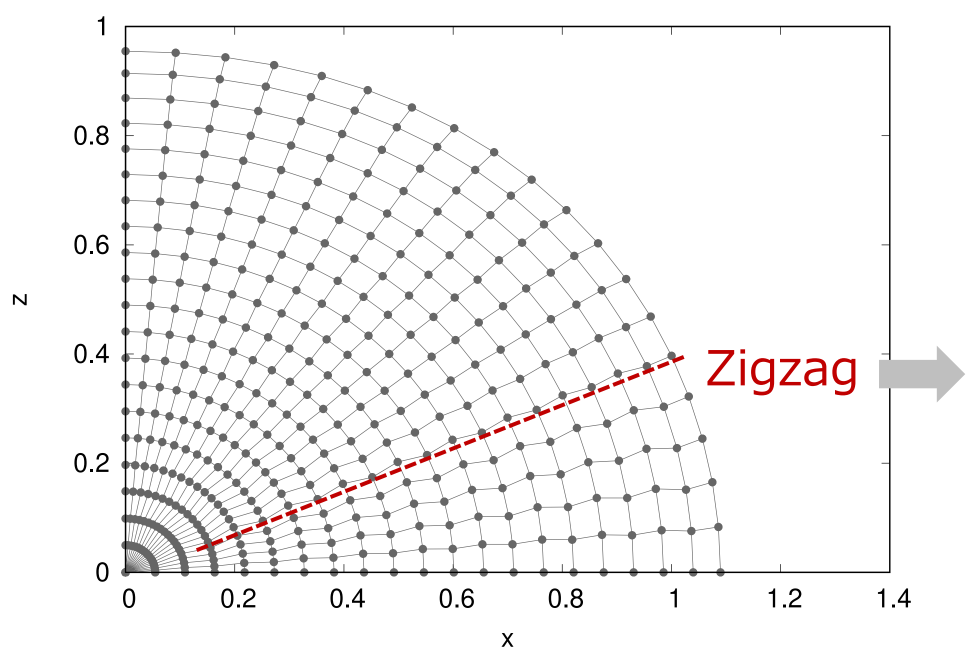
✓ Interpolation

Method: Remeshing

■ Remeshing

Dealing with problems due to mesh deformation
(due to numerical errors associated with finite differences)

1. Create a new smooth grid
2. Redistribute conserved quantities on the new grid



✓ Inaccuracy
✓ Non-convergence

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  - ✓ Configuration (Lagrangian formulation)
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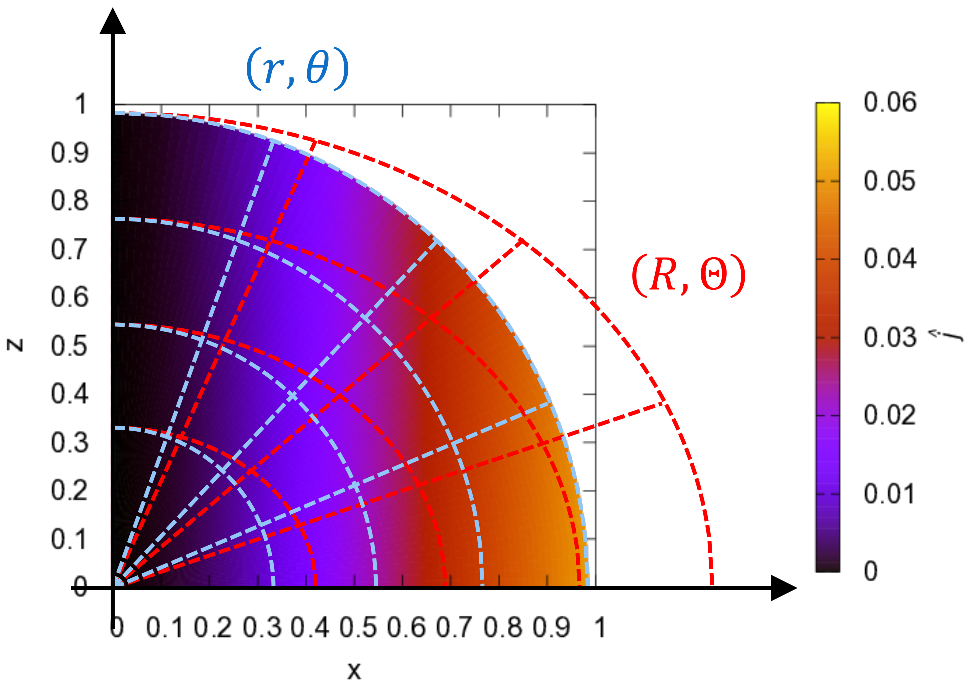
Model

- EOS: Barotrope
  - ✓ Isobaric and isopycnic surfaces coincide

$$P = K_0 \rho^{1+\frac{1}{N}} \qquad (N = 1.5)$$

- Rotation law

For $r \sin\theta \gg 1$, j-constant law (Eriguchi & Mueller 1985)



$$j(r, \theta) = \frac{j_0 (r \sin\theta)^2}{1 + (r \sin\theta)^2}$$

Results (Barotrope)

- Models with different rotation rates

Intermediate

$$\frac{R_\mathrm{p}}{R_\mathrm{eq}} = 0.743$$

$$\frac{T}{|W|} = 5.46\times10^{-2}$$

Rapid

$$\frac{R_\mathrm{p}}{R_\mathrm{eq}} = 0.380$$

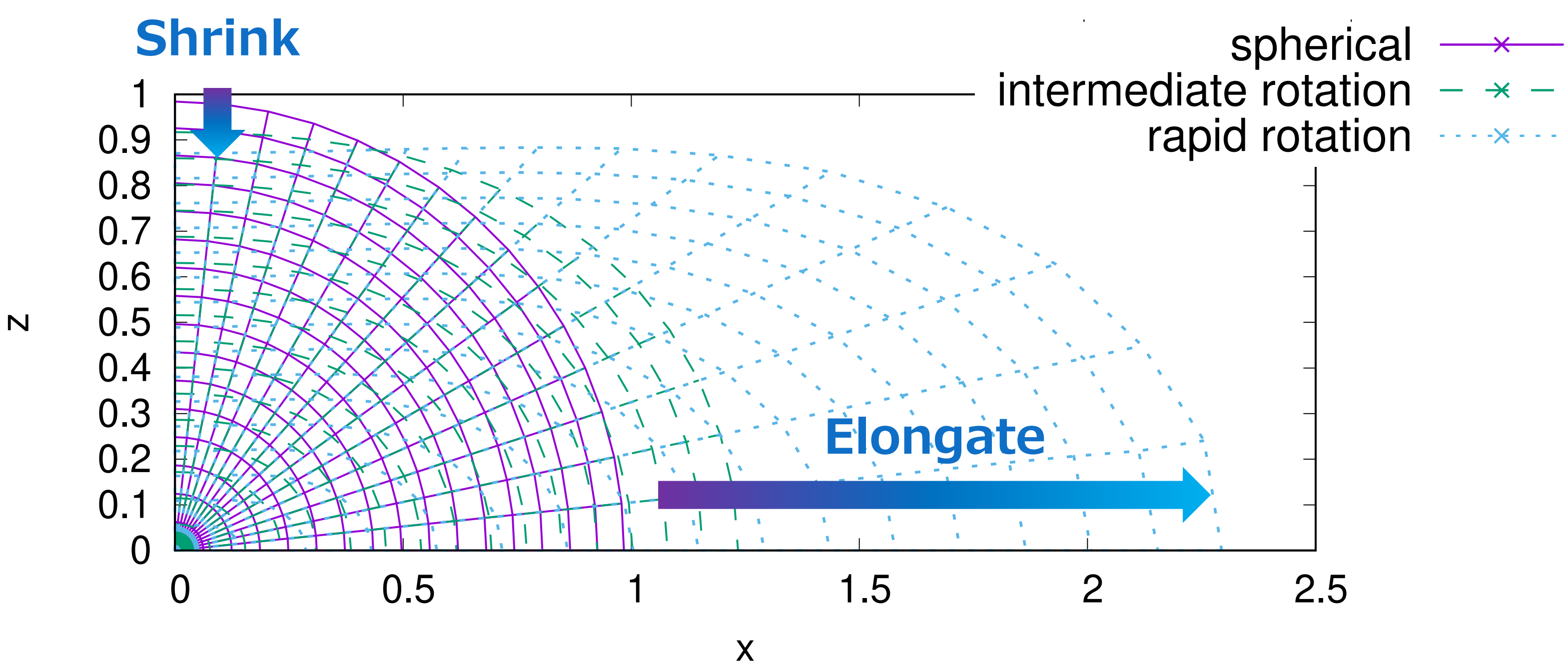
$$\frac{T}{|W|} = 0.168$$

Gravitational energy :

$$W \equiv \frac{1}{2}\int \rho\phi \, dV$$

Rotational energy :

$$T \equiv \frac{1}{2}\int \rho\Omega^2 (R\sin\Theta)^2 \, dV$$

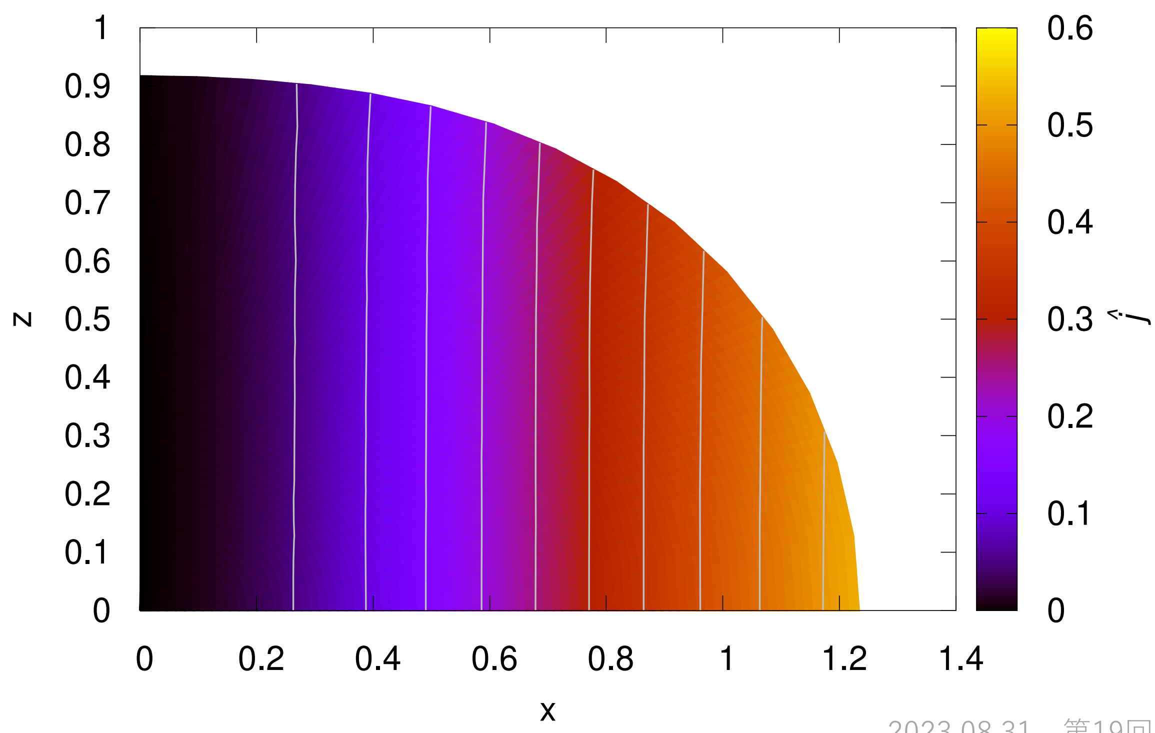


Results (Barotrope)

- Angular momentum distribution
  - ✓ Angular momentum is determined solely by distance from the axis of rotation

$N = 1.5$
$(N_r, N_\theta) = (16, 16)$

Intermediate rotation model



Results (Barotrope)

- Convergence of accuracy

  Our finite-difference scheme is of second-order accuracy.

$$V_c = \frac{|3U + W + 2T|}{|U| + |W| + |T|}$$

Gravitational energy :

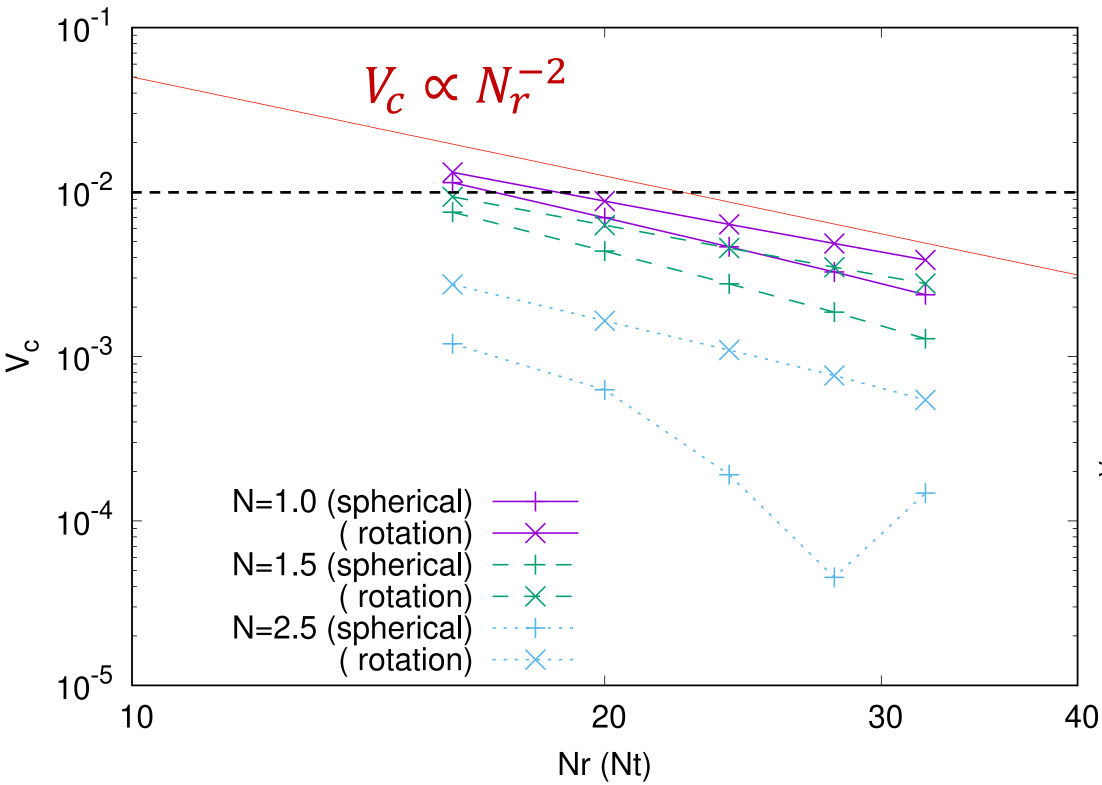
$$W \equiv \frac{1}{2}\int \rho\phi \, dV$$

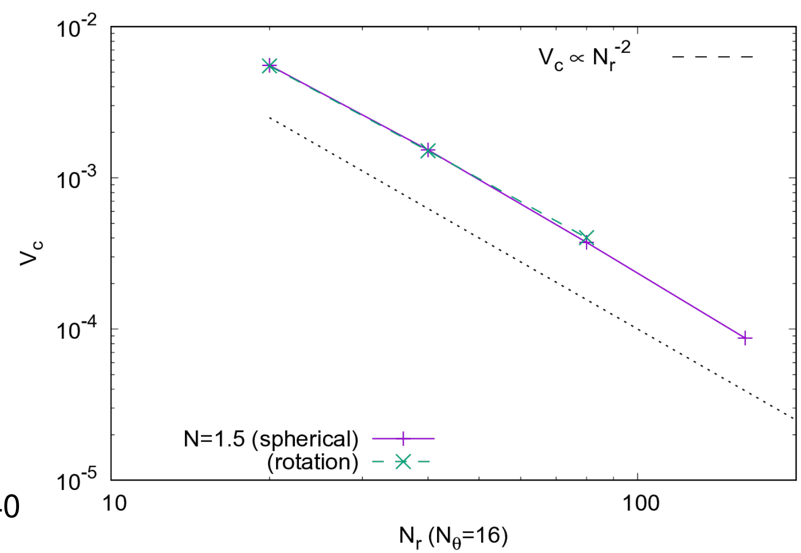
Rotational energy :

$$T \equiv \frac{1}{2}\int \rho\Omega^2 (R\sin\Theta)^2 \, dV$$

Integrated pressure :

$$U \equiv \int P \, dV$$





Summary

A new method for calculating the hydrostatic equilibrium structure of stars in 2D using a Lagrangian formulation

Configuration

- ✓ Lagrangian formulation
- ✓ W4 method

Self-gravity

- ✓ Spectral method

Remeshing

- ✓ Interpolation

Future Work

- ✓ Multilayered stars
- ✓ Introduction to various physics
- ✓ Application to general relativity (Okawa et al. 2022)